Comment on "Improved bounds on entropic uncertainty relations"

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We provide an analytical proof of the entropic uncertainty relations presented by de Vicente and Sánchez-Ruiz in [Phys. Rev. A **77**, 042110 (2008)] and also show that the replacement of Eq. (27) by Eq. (29) in that reference introduces solutions that do not take fully into account the contraints of the problem, which in turn leads to some mistakes in their treatment.

Consider two observables A and B, nondegenerate, with discrete spectra and complete orthonormal sets of eigenvectors $\{|a_i\rangle\}_{i=1}^N$ and $\{|b_j\rangle\}_{j=1}^N$, respectively. Denote by $p_i(A) = |\langle a_i|\Psi\rangle|^2$ the probabilities for the outcomes of observable A (and analogously for B) when the system is in the (pure) quantum state $|\Psi\rangle$. Let $c = \max_{i,j} |\langle a_i|b_j\rangle| \in \left[\frac{1}{\sqrt{N}}; 1\right]$ be the so called overlap between the observables. Maassen and Uffink (MU) [1] proved a non trivial universal lower bound for the sum

$$H(A) + H(B) \geq -2\ln c = \mathcal{B}_{MU}, \qquad (1)$$

where $H = -\sum_i p_i \ln p_i$ denotes Shannon entropy. This entropic uncertainty relation (EUR) has also been proved by Bialynicki-Birula and Mycielski [2] in the special case when the observables are conjugated, namely $\langle a_i|\Psi\rangle$ and $\langle b_i|\Psi\rangle$ are linked by a Fourier transform; then the bound is sharp in the sense that there exists a state $|\Psi\rangle$ for which the inequality is saturated, with $H(A) + H(B) = \ln N$.

In Ref. [3], de Vicente and Sánchez-Ruiz present an improvement of the MU-EUR (1), showing numerically that $H(A) + H(B) \geq \mathcal{B}_{VS}$ where the bound reads

$$\mathcal{B}_{VS} = \begin{cases} -2\ln c & \text{if } 0 < c \leq \dfrac{1}{\sqrt{2}} \\[2mm] \mathcal{H}_1(c) & \text{if } \dfrac{1}{\sqrt{2}} \leq c \leq c^* \\[2mm] \mathcal{F}(c) & \text{if } c^* \leq c \leq 1 \end{cases} \qquad (2)$$

with

$$\mathcal{F}(c) = -(1+c)\ln\left(\frac{1+c}{2}\right) - (1-c)\ln\left(\frac{1-c}{2}\right) \qquad (3)$$

and

$$\mathcal{H}_1(c) = -P_A \ln P_A - (1-P_A)\ln(1-P_A) \\ -P_B \ln P_B - (1-P_B)\ln(1-P_B) \qquad (4)$$

where

$$P_A \equiv \cos^2\alpha, \quad P_B \equiv \cos^2(\theta-\alpha), \quad c \equiv \cos\theta, \qquad (5)$$

and α is a (numerical) solution of the equation

$$0 = \sin(2\alpha)\ln\left(\frac{1+\cos(2\alpha)}{1-\cos(2\alpha)}\right) + \\ + \sin(2(\alpha-\theta))\ln\left(\frac{1+\cos(2(\alpha-\theta))}{2(1-\cos^2(\alpha-\theta))}\right) \qquad (6)$$

where $\alpha \neq \theta/2$ and $\alpha \neq \theta/2 + \pi/4$ in order to specify $P_A \neq P_B$. The approximate value of c^* is determined numerically in [3].

We show that the replacement of Eq. (27) of Ref. [3] by Eq. (6), via the change of variables (5), introduces solutions that do not take fully into account the constraints of the problem. This could potentially lead to erroneous conclusions. In the sequel we provide an analytical proof of such results, discussing in detail all possible cases. At the end of the Comment we give some concluding remarks.

The mechanism proposed in Ref. [3] to improve the bound (1) introduces the Landau–Pollak inequality (LPI)

$$\arccos\sqrt{P_A} + \arccos\sqrt{P_B} \geq \arccos c, \qquad (7)$$

where $P_I = \max_i p_i(I)$ for $I = A, B$, in two steps. First, for $I = A$ and B, minimize Shannon entropy $H(I)$ subject to a fixed maximum probability P_I, which leads to the minimal entropies $H_{\min}(P_I)$; second, search for the infimum of

$$\mathcal{M}(P_A, P_B) = H_{\min}(P_A) + H_{\min}(P_B), \qquad (8)$$

over the possible P_A and P_B, subject to the LPI. According to Ref. [3], the normalized probability distribution that minimizes $H(I)$ subject to fixed P_I is $(\underbrace{P_I, \ldots, P_I}_{M_I}, 1 - M_I P_I, 0, \ldots, 0)$, where M_I is a positive integer such that

$$\frac{1}{M_I + 1} < P_I \leq \frac{1}{M_I}. \qquad (9)$$

Then, one has

$$H_{\min}(P_I) = -M_I P_I \ln P_I - (1 - M_I P_I)\ln(1 - M_I P_I) \qquad (10)$$

for $I = A$ and B, and the minimization of $\mathcal{M}$ is restricted by the inequality constraints (7) and (9). We present rigorous solutions to the problem, for different cases.

Case $P_A \neq \frac{1}{M_A}$ and $P_B \neq \frac{1}{M_B}$:
The minimization is solved introducing the Lagrangian

$$\sum_{I=A,B} \left[H_{\min}(I) + \mu_I \left(P_I - \frac{1}{M_I} \right) + \nu_I \left(\frac{1}{M_I+1} - P_I \right) \right] +$$

$\lambda \left(\arccos c - \arccos \sqrt{P_A} - \arccos \sqrt{P_B} \right)$ with Lagrange parameters $\mu_A, \mu_B, \nu_A, \nu_B$ and λ. Deriving with respect to P_A and P_B and using Karush–Khun–Tucker necessary conditions for a minimum, one has

$$-M_I \ln \frac{P_I}{1 - M_I P_I} + \frac{\lambda}{2\sqrt{P_I (1 - P_I)}} + \mu_I - \nu_I = 0, \quad (11)$$

$$\mu_I \left(P_I - \frac{1}{M_I} \right) = 0, \quad \nu_I \left(\frac{1}{M_I + 1} - P_I \right) = 0, \quad (12)$$

$$\lambda \left(\arccos c - \arccos \sqrt{P_A} + \arccos \sqrt{P_B} \right) = 0 \quad (13)$$

with $\mu_I, \nu_I, \lambda \geq 0$, for $I = A$ and B.

Since the inequalities (9) are strict, from (12) one has $\mu_I = \nu_I = 0$. It is proved in Ref. [3] that $\lambda \neq 0$ (otherwise $P_I = 1/(M_I + 1)$); therefore, the LPI becomes an equality. Taking the cosine of this equality and using the constraints (9) one has

$$c = \sqrt{P_A P_B} - \sqrt{(1 - P_A)(1 - P_B)} \quad (14)$$

$$\leq \frac{1 - (M_A - 1)(M_B - 1)}{\sqrt{M_A M_B}}. \quad (15)$$

As M_I is a positive integer and $c > 0$, at least one M_I must be unity, and then

$$c \leq \frac{1}{\sqrt{\max (M_A, M_B)}}. \quad (16)$$

One can assume $M_A = 1$ and $M_B = M \geq 1$. Extracting λ from Eqs. (11) when $I = A$ and B, one gets

$$\sqrt{P_A (1 - P_A)} \ln \left(\frac{P_A}{1 - P_A} \right) =$$

$$M \sqrt{P_B (1 - P_B)} \ln \left(\frac{P_B}{1 - M P_B} \right). \quad (17)$$

These equations have *several solutions*. For example, a solution for $M = 1$ is given by $P_A = P_B = \frac{1+c}{2}$, making the function $\mathcal{F}$ given in Eq. (3) a possible candidate for a lower bound of the entropy sum. For $P_A \neq P_B$, Eqs. (14)–(17) do not have analytic solutions. At this stage, the authors in Ref. [3] perform the change of variables (5) and solve numerically Eq. (6), instead of Eq. (17) with $M = 1$, proposing the function $\mathcal{H}_1$ given in Eq. (4) as a possible minimum in a range $0 < c \leq c^*$. This is the critical point that motivates this Comment.

We will present below a detailed analysis that exhibits the following facts, depending on the value of the overlap:

1. In the range $0 < c < \frac{1}{\sqrt{2}}$:

   • For $M = 1$ Eq. (17) has only the trivial solution $P_A = P_B$, leading to $\mathcal{F}$. The value of $\mathcal{H}_1$ reported in Ref. [3] corresponds to P_A and P_B outside the allowed interval [see Eqs. eqs. (9) and (14)]. In fact, Eq. (17) for $M = 1$ (together with (9) and (14)) have only the trivial solution $P_A = P_B$, leading to $\mathcal{F}$. This seems to open the way to improve the MU bound in this range.

   • However, the extremum attained at $\frac{1+c}{2}$ happens to be a *maximum* for $\mathcal{M}$. If this value lies below the MU bound, so is the minimum; but, for the range $c \in (c^\dagger; \frac{1}{\sqrt{2}})$ with $c^\dagger \approx 0.61$, the maximum is higher than $-2 \ln c$.

   • In fact P_A (resp. P_B) "lives" within a given interval and the minimum of $\mathcal{M}$ is attained *at the end points* of that interval. Moreover, this minimum is less than $-2 \ln c$, which analytically and rigorously proves the result of Ref. [3] in $0 < c \leq \frac{1}{\sqrt{2}}$.

2. In the range $\frac{1}{\sqrt{2}} \leq c \leq c^*$: solution $\frac{1+c}{2}$ still corresponds to a maximum for $\mathcal{M}(P_A, P_B)$. However, Eq. (17) (with $M = 1$) admits *two symmetrical solutions* yielding the same minimum $\mathcal{H}_1$. We prove analytically that the extremizing values of P_A and P_B satisfy the constraints (9) and (14). However, the value $\mathcal{H}_1(c)$ can be evaluated only numerically. The result given in [3] is then confirmed in this range. In passing, we prove that c^* is a solution of the transcendental equation

   $$c^* \ln \left(\frac{1 + c^*}{1 - c^*} \right) = 2 \quad \Leftrightarrow \quad c^* \operatorname{argtanh} c^* = 1, \quad (18)$$

   then $c^* \approx 0.834$ as found in Ref. [3].

3. In the range $c^* < c \leq 1$: only the solution $\frac{1+c}{2}$ remains, as observed in Ref. [3]. Moreover, it corresponds there to a minimum. We justify this *analytically*, confirming the result of Ref. [3]

The proofs are as follows. First, we rewrite Eq. (14) as $c + \sqrt{(1 - P_A)(1 - P_B)} = \sqrt{P_A P_B}$. As both sides are positive, they can be squared without further ado leading to a quadratic equation in $\sqrt{1 - P_B}$ whose only allowed solution is

$$\sqrt{1 - P_B} = \sqrt{P_A(1 - c^2)} - c\sqrt{1 - P_A}. \quad (19)$$

Solving for P_B for given c gives

$$P_B(P_A) = \left(\sqrt{(1 - c^2)(1 - P_A)} + c\sqrt{P_A} \right)^2. \quad (20)$$

We realize that P_A, apart from lying between $\frac{1}{2}$ and 1 (from (9) since $M_A = 1$), is constrained to be larger than c^2 (from the positivity of (19)). Furthermore, the bounds

(9) for $I = B$ applied to (19) yield additional constraints on P_A. Summarizing, when $M = 1$, we get

$$\left(P_A^-; P_A^+\right) = \begin{cases} \left(\dfrac{1}{2}; \dfrac{(c + \sqrt{1 - c^2})^2}{2}\right) & \text{if } 0 < c < \dfrac{1}{\sqrt{2}} \\ \left(c^2; 1\right) & \text{if } \dfrac{1}{\sqrt{2}} \le c \le 1. \end{cases}$$
$$(21)$$

Next, we consider $\mathcal{M}_M(P_A) = \mathcal{M}(P_A, P_B(P_A))$. The goal is to study its behavior versus P_A so as to determine its minimum. To such end we compute successive derivatives of $\mathcal{M}_M$, with the help of some auxiliary functions. Noting that, for given c, $\frac{dP_B}{dP_A} = -\frac{\sqrt{P_B(1-P_B)}}{\sqrt{P_A(1-P_A)}}$ we obtain $\mathcal{M}'_M(P_A) = \frac{\mathcal{E}_M(P_A)}{\sqrt{P_A(1-P_A)}}$, where

$$\mathcal{E}_M(P_A) = M\sqrt{P_B(1-P_B)} \ln\left(\frac{P_B}{1 - MP_B}\right)$$
$$- \sqrt{P_A(1 - P_A)} \ln\left(\frac{P_A}{1 - P_A}\right). \quad (22)$$

has the same sign as $\mathcal{M}'_M$. It is obvious that setting $\mathcal{E}_M = 0$ solves (17). In the sequel, let us restrict ourselves to the case $M = 1$ (we will confirm later that the cases $M > 1$ need not to be considered). We demonstrate now that $\mathcal{E}_1$ has only four types of behavior. Its derivative writes as $\mathcal{E}'_1(P_A) = -\frac{\mathcal{K}(P_A)}{2\sqrt{P_A(1-P_A)}}$, where

$$\mathcal{K}(P_A) = (1 - 2P_B) \ln\left(\frac{P_B}{1 - P_B}\right)$$
$$+ (1 - 2P_A) \ln\left(\frac{P_A}{1 - P_A}\right) + 4, \quad (23)$$

has the opposite sign of $\mathcal{E}'_1$. We will see that $\mathcal{K}$ has always the same behavior versus P_A independently of c: it increases up to a maximum and then decreases. This behavior, together with the sign of the maximum of $\mathcal{K}$, completely determines the shape of $\mathcal{E}_1$. The derivative of $\mathcal{K}$ is $\mathcal{K}'(P_A) = \frac{\mathcal{N}(P_A)}{\sqrt{P_A(1-P_A)}}$, where

$$\mathcal{N}(P_A) = 2\sqrt{P_B(1-P_B)} \ln\left(\frac{P_B}{1 - P_B}\right) - \frac{1 - 2P_B}{\sqrt{P_B(1-P_B)}}$$
$$- 2\sqrt{P_A(1-P_A)} \ln\left(\frac{P_A}{1 - P_A}\right) + \frac{1 - 2P_A}{\sqrt{P_A(1-P_A)}}$$

has the same sign as $\mathcal{K}'$. Finally, the derivative of $\mathcal{N}$ is $\mathcal{N}'(P_A) = \frac{dP_B}{dP_A} \frac{(1 - 2P_B)\mathcal{R}(P_B) + 4}{\sqrt{P_B(1-P_B)}} - \frac{(1 - 2P_A)\mathcal{R}(P_A) + 4}{\sqrt{P_A(1-P_A)}}$, where $\mathcal{R}(x) = \ln\left(\frac{x}{1-x}\right) + \frac{1 - 2x}{2x(1-x)}$. From the negativity of $\mathcal{R}(x)$ for $x \in (\frac{1}{2}; 1)$ and of $\frac{dP_B}{dP_A}$, we conclude that $\mathcal{N}' < 0$. Summing up, for any c, $\mathcal{N}$ is continuous, strictly decreases in $(P_A^-; P_A^+)$, and has only one root: $\mathcal{N}\left(\frac{1+c}{2}\right) = 0$. As a result, $\mathcal{K}$ increases with P_A in the interval $(P_A^-; \frac{1+c}{2})$ and decreases in $(\frac{1+c}{2}; P_A^+)$. Furthermore, we notice that $\lim\limits_{P_A \to P_A^-} \mathcal{K}(P_A) = \lim\limits_{P_A \to P_A^+} \mathcal{K}(P_A)$ due to the fact that when

$P_A \to P_A^-$ then $P_B \to P_A^+$, and vice versa. With respect to the sign of $\mathcal{K}$, we show that only three situations arise:

1. when $c \in \left(0; \frac{1}{\sqrt{2}}\right)$, the maximum of $\mathcal{K}$, given by $\mathcal{K}\left(\frac{1+c}{2}\right) = -2c \ln\left(\frac{1+c}{1-c}\right) + 4$, is positive. Besides, the value of $\mathcal{K}$ at the end points, given by $\lim\limits_{P_A \to P_A^\pm} \mathcal{K}(P_A) = -2c\sqrt{1 - c^2} \ln\left(\frac{1 + 2c\sqrt{1-c^2}}{1 - 2c\sqrt{1-c^2}}\right) + 4$, decreases with c from 4 to $-\infty$ and thus can have either sign.

2. when $c \in \left(\frac{1}{\sqrt{2}}; c^*\right)$, the maximum of $\mathcal{K}$ is also positive, while $\lim\limits_{P_A \to P_A^\pm} \mathcal{K}(P_A) = -\infty$. We can determine the value of c^* precisely when the maximum $\mathcal{K}$ becomes zero, arriving at Eq. (18).

3. when $c \in (c^*; 1)$, the maximum of $\mathcal{K}$ is negative and thus $\mathcal{K} < 0$ for all P_A.

Going back to functions $\mathcal{E}_1$ and $\mathcal{M}_1$ we conclude that:

1. when $c \in \left(0; \frac{1}{\sqrt{2}}\right)$, if $\mathcal{K}(P_A^\pm) \ge 0$, $\mathcal{E}'_1 \le 0$ and thus $\mathcal{E}_1$ is strictly decreasing; on the contrary, if $\mathcal{K}(P_A^\pm) < 0$, $\mathcal{E}_1$ increases, decreases, and again increases. In both cases, $\lim\limits_{P_A \to P_A^-} \mathcal{E}_1(P_A) =$
   $$- \lim\limits_{P_A \to P_A^+} \mathcal{E}_1(P_A) = \frac{1 - 2c^2}{4} \ln\left(\frac{\sqrt{1 - c^2} + c}{\sqrt{1 - c^2} - c}\right) > 0;$$
   hence $\mathcal{E}_1 = 0$ has only one solution, given by $P_A = \frac{1+c}{2}$. This justifies that the value of $\mathcal{H}_1$ is computed from values of P_A and P_B that do not satisfy constraints eq. (9) in this range. Moreover, $\mathcal{E}_1$ changes from positive to negative sign at $\frac{1+c}{2}$, implying that: (i) the extremum of $\mathcal{M}_1$ at $\frac{1+c}{2}$ corresponds in fact to a maximum as we previously claimed, and (ii) the minimum of $\mathcal{M}_1$ would be attained at the end points $P_A^\pm$. As a conclusion, $\mathcal{M}_1$ is lower-bounded by $\lim\limits_{P_A \to P_A^\pm} \mathcal{M}_1(P_A)$ given by

$$\mathcal{M}_{\text{inf}} = -\frac{1 + 2c\sqrt{1-c^2}}{2} \ln\left(\frac{1 + 2c\sqrt{1-c^2}}{4}\right)$$
$$- \frac{1 - 2c\sqrt{1-c^2}}{2} \ln\left(\frac{1 - 2c\sqrt{1-c^2}}{4}\right). \quad (24)$$

We now define the difference of MU-bound to this infimum: $\Delta_{\mathcal{M}_{\text{inf}}}(c) = \mathcal{B}_{MU}(c) - \mathcal{M}_{\text{inf}}(c)$. The derivative of $\Delta_{\mathcal{M}_{\text{inf}}}$ with respect to c is $\frac{1 - 2c^2}{\sqrt{1 - c^2}} \left[\ln\left(\frac{1 + 2c\sqrt{1-c^2}}{1 - 2c\sqrt{1-c^2}}\right) - \frac{2\sqrt{1-c^2}}{c(1 - 2c^2)}\right]$ which can be analytically proved easily to be always negative in the range $c \in (0; \frac{1}{\sqrt{2}})$. This implies that $\Delta_{\mathcal{M}_{\text{inf}}}(c)$ is decreasing, with the lowest difference given by $\lim\limits_{c \to \frac{1}{\sqrt{2}}^-} \Delta_{\mathcal{M}_{\text{inf}}}(c) = \ln 2 > 0$. This analytically proves that $\mathcal{M}_{\text{inf}} < \mathcal{B}_{MU}$: it is impossible to improve the MU-EUR in the range $0 < c < \frac{1}{\sqrt{2}}$. This

confirms the result of Ref. [3]. Notice that studying what happens for $M > 1$ or for $P_I = \frac{1}{M_I}$ is then not necessary.

2. when $c \in \left(\frac{1}{\sqrt{2}}; c^*\right)$, $\mathcal{E}_1$ increases, decreases, and again increases, with $\lim_{P_A \to P_A^-} \mathcal{E}_1(P_A) = -\lim_{P_A \to P_A^+} \mathcal{E}_1(P_A) = -c\sqrt{1-c^2}\ln\left(\frac{c^2}{1-c^2}\right) < 0$; hence $\mathcal{E}_1 = 0$ has now three solutions: $\frac{1+c}{2}$, corresponding to a maximum of $\mathcal{M}_1$ ($\mathcal{E}_1$ locally decreases), and other two giving the same minimum for $\mathcal{M}_1$ (by symmetry). The minimum value of $\mathcal{M}_1$ in this range is denoted as $\mathcal{H}_1$ in Ref. [3], where it is obtained after solving numerically for α in Eqs. (5)–(6). The same result is obtained here directly from (17) and (20), taking care of the constraints (21) for P_A (notice that only $M = 1$ has to be considered as (16) enforces $M \le \frac{1}{c^2}$). We also numerically confirm that $\mathcal{H}_1(c) > -2\ln c$, thus giving the possibility of improving MU-EUR in this range (see the cases $P_I = 1/M_I$ below).

3. when $c \in (c^*; 1)$, $\mathcal{E}_1$ always increases, the limiting values at the end points are the same as in 2 above, but the unique root $\frac{1+c}{2}$ corresponds to a minimum: $\mathcal{M}_1\left(\frac{1+c}{2}\right) = \mathcal{F}(c)$ given in Eq. (3) (notice that only $M = 1$ has to be considered since $M \le \frac{1}{c^2}$). Consider now the difference $\Delta_{\mathcal{F}}(c) = \mathcal{B}_{MU}(c) - \mathcal{F}(c)$, whose derivative is $\frac{2}{c}\left(c\ln\left(\frac{1+c}{1-c}\right) - 2\right) > 0$ in the range $c > c^*$. Thus $\Delta_{\mathcal{F}}(c)$ increases; since $\Delta_{\mathcal{F}}(1) = 0$, then $\Delta_{\mathcal{F}}(c) < 0$ in this range and therefore we analytically prove that $\mathcal{F} > \mathcal{B}_{MU}$, as observed in [3]. Again the MU-EUR can possibly be improved in this range (see the cases $P_I = 1/M_I$ below).

What happens in the particular cases $c = \frac{1}{\sqrt{2}}$ and $c = c^*$, follows from the continuity of the functions involved.

Cases $P_A = \frac{1}{M_A}$ and/or $P_B = \frac{1}{M_B}$:
In Ref. [3] the authors check the case $P_I = \frac{1}{M_I}$ for $I = A$ or B and find as a possible minimum for $\mathcal{M}(P_A, P_B)$

$$\mathcal{G}(c) = -c^2\left\lfloor\frac{1}{c^2}\right\rfloor\ln c^2 - \left(1 - c^2\left\lfloor\frac{1}{c^2}\right\rfloor\right)\ln\left(1 - c^2\left\lfloor\frac{1}{c^2}\right\rfloor\right),\tag{25}$$

where $\lfloor\cdot\rfloor$ indicates integer part (floor). They base their procedure on the equivalent of Eqs. (11) and claim that M must be either 1 or 2, assuming the nonnegativity of the Lagrange multipliers. As far as we understand, this reasoning seems erroneous, i.e. the multiplier μ_I corresponding to the "equality constraint" $P_I = \frac{1}{M_I}$ is not necessarily nonnegative, as should be for "inequality constraints".

First of all, as already mentioned, the MU-EUR could be improved only when $c > \frac{1}{\sqrt{2}}$. For $P_A = \frac{1}{M_A}$ and $P_B \ne \frac{1}{M_B}$ (or the contrary, exchanging the roles of observables A and B), we have for $I = B$ the same Eqs. (11) and (12); also Eq. (13) remains valid. One still has $\mu_B = \nu_B = 0$, and thus $\lambda \ne 0$. The LPI remains saturated, which means that Eqs. (14), (16), (19) and (20) are still valid. Thus, the constraint $P_A > c^2$ due to Eq. (19) enforces $P_A = 1$, thus $M_A = 1$. As a consequence, Eq. (20) gives $P_B = c^2$, which means also that, from (9), $M_B = \lfloor\frac{1}{c^2}\rfloor$. In turn, one arrives at the function $\mathcal{G}$ given above, valid only for $c > \frac{1}{\sqrt{2}}$. This also entails that $\lfloor\frac{1}{c^2}\rfloor = 1$ and then $\mathcal{G} = -c^2\ln c^2 - (1 - c^2)\ln(1 - c^2)$ which corresponds, as it should, to the Shannon entropy of the probability distribution $(c^2, 1 - c^2, 0, \ldots, 0)$. One can numerically prove that $\mathcal{G} > \mathcal{H}_1$ and, analytically, that for $c \ge c^*$, $\mathcal{G} > \mathcal{F}$; therefore $\mathcal{G}$ does not correspond to the minimal $\mathcal{M}$.

Finally, we study the case $P_A = \frac{1}{M_A}$ and $P_B = \frac{1}{M_B}$. The sum of the minimum Shannon entropies is $\mathcal{M}(\frac{1}{M_A}, \frac{1}{M_B}) = \ln(M_A M_B)$. It is straightforward to see that the minimal $\mathcal{M}$ is: 0 if $c = 1$ or $\ln M$ if $\frac{1}{\sqrt{M}} \le c < \frac{1}{\sqrt{M-1}}$ where $M = 2, 3, 4, \ldots$. Clearly these bounds are non-optimal.

Summing up, we revisit analytically the full resolution of the problem presented in Ref. [3] that deals with the uncertainty related to the measurement of two discrete quantum observables, using as measure the sum of Shannon entropies associated to both distributions constrained by the Landau-Pollak inequality. De Vicente and Sánchez-Ruiz show in [3] that the Maassen-Uffink bound can be improved using this constraint when the overlap c between observables is in the range $(1/\sqrt{2}; 1)$; we confirm analytically this result. Our central contributions were to provide an analytical proof of the non-improvement of the bound when c is in the range $(0, 1/\sqrt{2})$, and the analytical proof that $\mathcal{F}$ is indeed a minimum of the entropy sum $\mathcal{M}$ for c in the range $(c^*; 1)$. Additionally, we obtained the value of c^* from an analytical expression, given in Eq. (18). We detected a mistake in the VS-treatment of the constrained extremization problem: the function $\mathcal{H}_1$ was computed for solutions of Eq. (6) that do not take into account the whole set of restrictions on the pertinent probabilities. This seemed to open the possibility of improving Maassen–Uffink bound in the range $c \in (0; \frac{1}{\sqrt{2}})$. But in fact, we rigorously show that it is impossible to improve MU-EUR with the LPI in this range.

Moreover, let us comment that the function $\mathcal{F}(c)$ can be interpreted as half the Jensen–Shannon divergence between the pure states $|a_i\rangle\langle a_i|$ and $|b_j\rangle\langle b_j|$, for which the overlap is maximum [4]. An interesting future research is to exploit this relationship for establishing new entropic uncertainty relations.

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